Formal Proofs for Quantifiers

∃ **Intro** and ∃ **Elim**

William Starr

04.23.09

Announcements

For 04.23

1. HW12 will be due on Thursday 04.30
   - HW12 is a practice final exam
   - It is up on Sakai
2. There is no HW13!
3. **Final Exam**:
   - Take-home portion given out on Thurs 04.30
   - Take-home portion due at in-class final exam on Weds 05.13 (8-11am)

Outline

∀ **Elim**

Universal Elimination

Universal Elimination (Official Version, Informal Step)

If you have $\forall x\, S(x)$, you may infer $S(c)$, provided that ‘c’ refers to an object in the domain of discourse

∀ Elim

	$\forall x\, S(x)$
	⋮
▷	$S(c)$

EXAMPLE:

1	$\forall x\,(\mathrm{Tet}(x) \land \mathrm{Small}(x))$	
2	$\mathrm{Tet}(c) \land \mathrm{Small}(c)$	∀ **Elim**: 1

∀ Intro
Versions 1 & 2

∀ Intro Version 1 (Formal Method of Proof)

	$\boxed{c}$
	⋮
	$P(c)$
▷	$\forall x\, P(x)$

c must not occur outside of the subproof where it is introduced
That is, c must be arbitrary

- Read the boxed 'c' as *let c be an arbitary member of the domain of discourse*

∀ Intro Version 2 (Formal Method of Proof)

	$\boxed{c}\ A(c)$
	⋮
	$B(c)$
▷	$\forall x\,(A(x) \rightarrow B(x))$

c must not occur outside of the subproof where it is introduced
That is, c must be arbitrary

- Read the boxed 'c' as *let c be an arbitary member of the domain of discourse*

∀ Intro
An Example

1	$\forall x\,[Tet(x) \rightarrow \neg Cube(x)]$	
2	$\forall x\,[\neg Cube(x) \rightarrow Small(x)]$	
3	$\boxed{c}\ Tet(c)$	
4	$Tet(c) \rightarrow \neg Cube(c)$	∀ **Elim**: 1
5	$\neg Cube(c)$	→ **Elim**: 3,4
6	$\neg Cube(c) \rightarrow Small(c)$	∀ **Elim**: 2
7	$Small(c)$	→ **Elim**: 5,6
8	$\forall x\,[Tet(x) \rightarrow Small(x)]$	∀ **Intro**: 3-7

Hint: working backwards is very helpful w/∀ **Intro**

Overview
This Section

- For existential quantification, we've learned one informal inference step & one informal proof method:
  1. Existential Introduction
  2. The Method of Existential Elimination
- Today, we'll learn the formal counterparts of these informal principles
- But first, let's review the informal principles

Informal Steps & Methods
With the Existential Quantifier

Existential Introduction (Official Version)

From $S(n)$ you can infer $\exists x\, S(x)$, provided 'n' names an individual in the domain of discourse

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

An Example

Using Existential Introduction and Elimination

$\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$

$\exists x\,\mathsf{Cube}(x)$

$\exists x\,\mathsf{Small}(x)$

Proof: We'll start by using existential elimination on premise two: let c be some cube. From premise 1 it follows by universal elimination that $\mathsf{Cube}(c) \rightarrow \mathsf{Small}(c)$ By modus ponens, c must be small. But then it follows that something is small (by existential introduction).

- Note that I applied existential elimination before universal elimination

Existential Intro

From Informal to Formal

Existential Introduction (Official Version, Informal Step)

If you have $\mathsf{S}(c)$, you may infer $\exists x\,\mathsf{S}(x)$, provided that 'c' refers to an object in the domain of discourse

∃ Intro

$\mathsf{S}(c)$

⋮

▷ $\exists x\,\mathsf{S}(x)$

EXAMPLE:

1	$\mathsf{Tet}(c)$	
2	$\exists x\,\mathsf{Tet}(x)$	∃ **Intro**: 1

Existential Elim

From Informal to Formal

Method of Existential Elimination

1. Given $\exists x\,\mathsf{S}(x)$, you may select a dummy name, say c, and assume $\mathsf{S}(c)$; whatever you can show from $\mathsf{S}(c)$ follows from the existential claim
2. c must be a new name, i.e. one not already in use in the context of your proof

∃ Elim

$\exists x\,\mathsf{S}(x)$

⋮

$\boxed{c}$ $\mathsf{S}(c)$

⋮

Q

▷ Q

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

The boxed c is read: *let c be an arbitrary individual such that...*

∃ Rules

An Example

1	$\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$	
2	$\exists x\,\mathsf{Cube}(x)$	
3	$\boxed{c}$ $\mathsf{Cube}(c)$	
4	$\mathsf{Cube}(c) \rightarrow \mathsf{Small}(c)$	∀ **Elim**: 1
5	$\mathsf{Small}(c)$	→ **Elim**: 3, 4
6	$\exists x\,\mathsf{Small}(x)$	∃ **Intro**: 5
7	$\exists x\,\mathsf{Small}(x)$	∃ **Elim**: 2, 3-6

∃ Rules
Another Example

Let's do exercise 13.12. In this chapter we are free to use **Taut Con** to justify proof steps involving only propositional connectives.

13.12

$$\begin{array}{|l} \forall x\,(\mathsf{Cube}(x) \lor \mathsf{Tet}(x)) \\ \exists x\,\neg\mathsf{Cube}(x) \\ \hline \exists x\,\mathsf{Tet}(x) \end{array}$$

∃ Rules
In Class Exercise

Construct a formal proof for the following argument.

$$\begin{array}{r|l} 1 & \forall x\,(\mathsf{Tet}(x) \rightarrow \mathsf{Medium}(x)) \\ 2 & \exists y\,\mathsf{Tet}(y) \\ \hline 3 & \exists x\,(\mathsf{Tet}(x) \land \mathsf{Medium}(x)) \end{array}$$

∃ Rules
More Practice

One more exercise in Fitch:

- Exercise 13.14

Mixing Quantifiers
A Pseudo-Proof

$$\begin{array}{r|l} 1 & \forall x\,\exists y\,\mathsf{Loves}(x, y) \\ \hline 2 & \exists y\,\forall x\,\mathsf{Loves}(x, y) \end{array}$$

Pseudo-Proof: The premise says that everyone likes someone or other. So let b be any boy, and g be a girl he loves. Since b was chosen arbitrarily, we may conclude by Univ. Intro. that $\forall x\,\mathsf{Loves}(x, g)$. By existential introduction, our conclusion follows.

- The crucial misstep in this proof was the claim that b was arbitrary
- By introducing g, a specific girl that b likes, b ceases to be arbitrary
  - Our proof then contains specific information about b: that he likes g, which is **not** true of everyone

Mixing Quantifiers

A Pseudo-Proof

1	$\forall x\, \exists y\, Loves(x, y)$	
2	$\boxed{b}$	
3	$\exists y\, Loves(b, y)$	$\forall$ **Elim**: 1
4	$\boxed{g}$ $Loves(b, g)$	
5	$Loves(b, g)$	Reit: 4
6	$Loves(b, g)$	$\exists$ **Elim**: 3,4-5 × × ×
7	$\forall x\, Loves(x, g)$	$\forall$ **Intro**: 6 × × ×
8	$\exists y\, \forall x\, Loves(x, y)$	$\exists$ **Intro**: 7

Lines 6 & 7: g occurs outside of subproof of introduction

Mixing Quantifiers

The Moral

Summary

1. **Universal Introduction**: the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced.
2. **Existential Elimination**: the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced
   - Above, g occurred outside of the subproof in which it was introduced!
3. It was nuanced to see exactly how to manage arbitrary constants in informal proofs
4. But in formal proofs, subproofs give us the resources to state these conditions precisely